Property of the low-lying states at the critical point of the phase transition in U(4) vibron model

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Abstract

We study the properties of the low-lying states at the critical point of the phase transition from U(3) to O(4) symmetry in the U(4) vibron model in detail. By analyzing the general characteristics and comparing the calculated results of the energy spectra and the E1, E2 transition rates in E(3) symmetry, in r^4 potential model and the finite boson number case in boson space, we find that the results in the r^4 potential demonstrates the characteristic of the classical limit at the critical point well and the E(3) symmetry over-predict the energy levels and under-predict the E1 and E2 transition rates of the states at the critical point. However, the E(3) symmetry may describe part of the properties of the system with boson number around 10 to 20. We also confirm that the ^{12}C+^{12}C system is an empirical evidence of the state at the critical point of the phase transition in the U(4) model when concerning the energies of the low-lying resonant states.

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Quantum phase transitions in mesoscopic system (system with a finite number of particles N), such as atomic nuclei [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], molecules [24, 25, 26], atomic clusters [27] and finite polymers have recently been attracting a lot of interests. The transition in these systems are among different shapes, geometric configurations, and modes of collective motions. For nuclei, it has been well known that there exists vibrational, γ-soft rotational, axially rotational, and other collective modes. The interacting bosons model (IBM, the simplest one is the U(6) model including s- and d-bosons) [6] has been shown to be successful in studying the properties of the low-lying collective states of even-even nuclei and the shape phase transitions (see for example, Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 23]). It has also been well known that there exhibits vibration and rotation in molecules. To characterize the relative motion of a dipole-deformation in the three dimensional space and describe the behavior of rotational and vibrational motions of molecules, the U(4) vibron model has been developed [24, 28] and applied to two-body (or two-cluster) systems such as diatomic molecules [28, 29], binary clusters [30, 31, 32, 33], $q\bar{q}$ mesons [34, 35], and so on. It has been shown that the U(4) model involves two dynamical symmetries, namely U(3) and O(4), and there exists a second order phase transition from U(3) to O(4) symmetry [36, 37]. For the properties of the states at the critical point of the vibrational to γ-soft rotational, the vibrational to axially rotational phase transition of nucleus, respectively, there have had thorough studies in theory (see, for example, Refs. [10, 38, 39, 40, 41, 42, 43, 44]) and quite a lot of empirical evidences have also been found (see, for instance, Refs. [45, 46]). For the properties of the states around the critical point of the U(3) and O(4) phase transition in the U(4) model, even though they can be described approximately with the E(3) symmetry, which has been discussed in some sense [37], they have not yet been investigated in detail. We will then study the low-lying energy spectrum and the E1, E2 transition rates of the low-lying states at the critical point of the phase transition in the U(4) vibration model in this paper.

In the U(4) model, elementary excitations are dipole p-bosons with spin and parity $J^\pi = 1^-$ and scalar s-bosons with $J^\pi = 0^+$. With assumptions that the total number of bosons and the angular momentum of the system are conserved, there are only two dynamical symmetry limits, U(3) and O(4). Accordingly, there exist two dynamical symmetry chains:

$$U(4) \supset U(3) \supset O(3), \qquad \text{(I)} \tag{1}$$

$$U(4) \supset O(4) \supset O(3). \qquad \text{(II)} \tag{2}$$

It has been shown that the U(3) symmetry corresponds to nonrigid ro-vibrations, while the O(4) symmetry represents rigid ro-vibrations [28]. A general Hamiltonian of the U(4) vibron model with only one- and two-body interactions being taken into account can be expressed in terms of the linear and quadratic invariant operators (Casimir operators) of all the subgroups

contained in the dynamical group chains.

To study the property of the phase transition between U(3) symmetry and O(4) symmetry, one starts usually from the simple Hamiltonian

$$\hat{H} = \varepsilon[(1-\eta)\hat{n} - \frac{\eta}{f(N)}\hat{D}\cdot\hat{D}]\,, \tag{3}$$

where ε is a scale parameter and it can be taken as one for convenience without any loss of generality. $\hat{n} = \sum_m p_m^\dagger p_m$ is the number operator of p-bosons, $\hat{D}_q^{(1)} = (s^\dagger\tilde{p} + p^\dagger\tilde{s})_q^{(1)}$ is the electric dipole operator, where $\tilde{s} = s$ and $\tilde{p}_m = (-1)^{1-m}p_{-m}$. $f(N)$ is a linear function of total boson number N. η is the control parameter. It is easy to show that such a Hamiltonian can be alternatively written as

$$\hat{H} = \varepsilon(1-\eta)C_{1U(3)} - \varepsilon\frac{\eta}{f(N)}C_{2O(4)} + \varepsilon\frac{\eta}{f(N)}C_{2O(3)}\,. \tag{4}$$

It is obvious that the system is in U(3) symmetry when $\eta = 0$, and in O(4) symmetry if $\eta = 1$. By varying $\eta \in [0,1]$, we can realize the U(3)-O(4) phase transition.

The classical limit corresponding to the Hamiltonian in Eq. (3) can be obtained by considering its expectation value of the coherent state [24]

$$|N;\mathbf{t}\rangle = (N!)^{-1/2}[(1-\mathbf{t}^*\cdot\mathbf{t})^{1/2}s^\dagger + \mathbf{t}\cdot\mathbf{p}^\dagger]^N|0\rangle\,, \tag{5}$$

where $\mathbf{t}$ is a complex three-dimensional vector, and its complex conjugate is denoted by $\mathbf{t}^*$. Then the classical Hamiltonian can be given as $H_{cl} = \langle N;\mathbf{ti} \mid \hat{H} \mid N;\mathbf{t}\rangle$. One can introduce canonical position and momentum variables $\mathbf{r}$ and $\mathbf{q}$ (they can be the ones in three dimension) by the transformation [24]

$$\mathbf{t} = (\mathbf{r} + i\mathbf{q})/\sqrt{2}\,, \qquad \mathbf{t}^* = (\mathbf{r} - i\mathbf{q})/\sqrt{2}\,. \tag{6}$$

So the classical potential is just the value of $H_{cl}(q,r)$ with $q = 0$, where $|\mathbf{r}| = r$ and $|\mathbf{q}| = q$ i.e.,

$$V(r) = H_{cl}(q=0,r)\,. \tag{7}$$

In the case of taking $f(N)$ in Eq. (3) to be $3N$, the energy potential corresponding to the Hamiltonian in Eq. (3) can be concretely expressed as

$$V(r) = N\Big[(1-\eta)\frac{r^2}{2} - \frac{\eta}{3}r^2(2-r^2)\Big]\,. \tag{8}$$

From Eq. (8), we could easily recognize that the classical potential for U(3) symmetry (with $\eta = 0$) is $V_{U(3)}(r) = \frac{N}{2}r^2$, while the classical potential for O(4) symmetry (with $\eta = 1$) is $V_{O(4)}(r) = -\frac{N}{3}r^2(2-r^2)$. By analyzing the stability of the system with potential in Eq. (8) we could find that the critical point of the quantum transition corresponds to the control parameter $\eta_c = \frac{3}{7}$. It is obvious that, as $\eta < \frac{3}{7}$, $V_{min}(r) = 0$; if $\eta > \frac{3}{7}$, $V_{min}(r) = -\frac{N}{48}\frac{(7\eta-3)^2}{\eta}$.

Furthermore, at the critical point with control parameter $\eta = \eta_c = \frac{3}{7}$, the energy potential can be explicitly written as

$$V_{cri}(r) = \frac{N}{7} r^4 \,, \tag{9}$$

and the Hamiltonian corresponding to Eq. (3) could be simplified to a Schrödinger equation

$$\hat{H}\Phi = \big[\frac{\hat{p}^2}{2m} + \frac{r^4}{7}\big]\Phi = E\Phi \,. \tag{10}$$

Looking through the characteristic of the potential at the critical point in Eq. (9), one can learn that it is quite flat around its bottom. To study the property of the states at the critical point of the phase transition in the U(4) model, at first one can go along the way taken in Ref. [38] and approximate the potential around the critical point to be a three-dimensional infinite square well

$$V(r) = \begin{cases} 0 \,, & r \leq r_W \,, \\ \infty \,, & r > r_W \,. \end{cases} \tag{11}$$

It is apparent that such a potential deviates from that of the E(5) symmetry of the transition from U(5) to O(6) symmetry in the IBM [38] only in dimension. Then the states generated from this potential can de denoted as the ones with the E(3) symmetry.

In such a situation, the Schrödinger equation is exactly solvable and the solution could be expressed as some Bessel function. The excitation energy can be given as

$$E_{n,L} = \frac{2B}{\hbar} k_{n,L}^2 \,, \tag{12}$$

with $k_{n,L}^2 = \frac{y_{n,L}}{x_W}$, where $y_{n,L}$ is the nth zero point of the Bessel function $J_{L+1/2}(z)$, B is a constant. We give in Fig. 1 the obtained energy spectrum of some of the low-lying states and in Table 1 the values of the excitation energies of the states, where the energy of the ground state is set to zero and all energies are normalized to the energy of the first excited state (with $L^\pi = 1_1^-$).

Then we solve the Eq. (10) numerically to discuss more practically the properties of the low-lying states at the critical point of the phase transition in the U(4) model. The obtained energy spectrum of the low-lying states is illustrated in Fig. 2 and the concrete values of some states' energies are listed in Table 1. To show the variation feature of the energy of low-lying states explicitly and make it easy to compare with experimental data, we display some of the energy ratios between some states in Fig. 3. Comparing the result in the E(3) symmetry with that obtained by solving the Schrödinger equation in potential of r^4 form, one can notice that, the energy of a state in the E(3) symmetry is higher than that given by the r^4 potential for the state with the same quantum number.

To investigate the properties of the states at the critical point more comprehensively, we also solved the engin-equation with Hamiltonian in Eq. (3) with $\eta = \frac{3}{7}$ directly in the cases of various boson numbers. The obtained results of the variation feature of the ratios

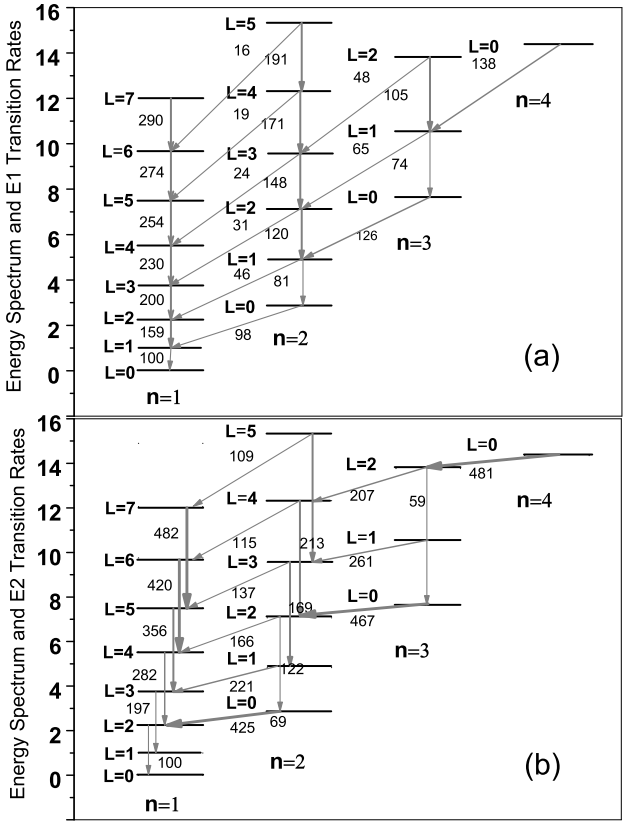

Figure 1: Calculated energy spectrum of the low-lying states in E(3) symmetry (with a potential of infinite square well) around the critical point of the phase transition U(3)-O(4) in the U(4) vibron model and the corresponding E1 (panel (a)), E2 (panel (b)) transition rates (the numbers close to the arrow are the value of the rate with normalization $B(\mathrm{E1}; 1^-_1 \to 0^+_1) = 100$, $B(\mathrm{E2}; 2^+_1 \to 0^+_1) = 100$, respectively).

of some low-lying states against the boson number and the comparison with those in the E(3) symmetry and r^4 potential are illustrated in Fig. 4. From Fig. 4, one can learn that, for the yrast states, the results given in the E(3) symmetry is quite close to that in the case of fewer bosons (for instance, around ten), but that determined by the r^4 potential is the asymptotic limit of the one in large boson number limit. It indicates that the one with the r^4 potential describes the behavior at the classical limit (or infinite boson number limit) more appropriately, and the E(3) symmetry over-predicts the energy levels. Such a feature is consistent with those of the states around the critical point of the phase transition from U(5) to O(6) in the U(6) model of IBM [40, 42].

With transition operators

$$\hat{T}(\mathrm{E1}) = -\vec{P} \cdot \vec{E}_0 = e_{1,eff} r \cos\theta \,, \tag{13}$$

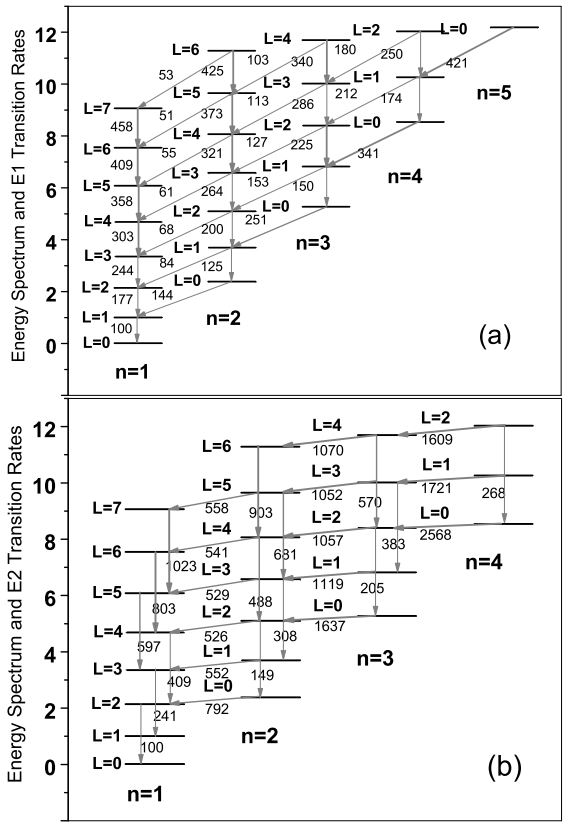

Figure 2: Calculated energy spectrum of the low-lying states at the critical point of the phase transition U(3)-O(4) (with a potential of r^4) in the U(4) vibron model and the corresponding E1 (panel (a)), E2 (panel (b)) transition rates (the numbers marked close to the arrow are the value of the rate with normalization $B(\mathrm{E1}; 1^-_1 \to 0^+_1) = 100$, $B(\mathrm{E2}; 2^+_1 \to 0^+_1) = 100$, respectively).

$$\hat{T}(\mathrm{E2}) = -\frac{1}{6}\sum_{ij} Q_{ij}\Big(\frac{\partial E_j}{\partial x_i}\Big)_0 = e_{2,eff} r^2(3\cos^2\theta - 1)\,, \tag{14}$$

where $Q_{ij} = q\langle 3x_i x_j - r^2\delta_{ij}\rangle$, we can determine the E1 and E2 transition rates $B(\mathrm{Ek}, L_i \to L_f) = \frac{1}{2L_i+1}|\langle n_f\, L_f\|\hat{T}(\mathrm{Ek})\|n_i\, L_i\rangle|^2$. The obtained results in case of the E(3) symmetry (i.e., with the potential of infinite square well) and those in potential of r^4 are displayed in Fig. 1, Fig. 2, respectively. We have also calculated the E1 and E2 transition rates with operator $\hat{T}(E1)_q = e_{1,eff}(s^\dagger\tilde{p} + p^\dagger s)^1 q$, $\hat{T}(E2)_\mu = e_{2,eff}(p^\dagger\tilde{p})^2\mu$, respectively, for the systems with different boson numbers. The calculated results of changing behavior of the ratios of some transition rates with respect to the boson number and the comparison with those in E(3) symmetry and r^4 potential are manifested in Fig. 5. Looking through Figs. 1, 2 and 5, one can notice that the E(3) symmetry under-predicts the E1 and E2 transitions rates between the low-lying states at the critical point with the r^4 potential in the U(4) model. Moreover,

Table 1: Excitation energies of the low-lying states at the critical point of the phase transition from U(3) to O(4) in the U(4) vibron model.

	$n=1$		$n=2$		$n=3$		$n=4$		$n=5$	
	E(3)	r^4	E(3)	r^4	E(3)	r^4	E(3)	r^4	E(3)	r^4
$L=0$	0.00	0.00	2.87	2.37	7.65	5.27	14.35	8.55	22.96	12.17
$L=1$	1.00	1.00	4.83	3.70	10.57	6.82	18.22	10.27	27.79	14.02
$L=2$	2.26	2.13	7.06	5.09	13.76	8.41	22.37	12.02	32.90	15.89
$L=3$	3.78	3.36	9.56	6.55	17.23	10.04	26.86	13.81	38.28	17.80
$L=4$	5.53	4.68	12.32	8.08	20.97	11.72	31.51	15.64	43.95	19.76
$L=5$	7.51	6.09	15.34	9.67	24.97	13.47	36.48	17.52	49.89	21.75
$L=6$	9.76	7.56	18.61	11.31	29.22	15.26	41.72	19.43	56.10	23.77
$L=7$	12.07	9.08	22.12	13.12	33.75	16.83	47.22	21.42	62.58	25.89
$L=8$	14.90	10.71	25.88	14.75	38.52	19.98	52.99	23.41	69.32	27.99

the E(3) symmetry can only describe the ratios of some transition rates of the system with boson number N around 10 to 20 with an exception of $\frac{B(E2:0_2\to 2_1)}{B(E2:2_1\to 0_1)}$, which deviates from both the result in E(3) symmetry and that in the r^4 potential in small N case but in large-N limit obviously approaches to the one in the r^4 potential. Therefore, the r^4 potential depicts excellently all the transition rate ratios as those energy ratios of the system in classical limit. The analyses of energy levels together and transition rates indicate that the results in the E(3) symmetry shows a nature finite N correction on those in the r^4 potential for the low-lying spectrum since the bottom of the potential in E(3) symmetry is flatter than that in the r^4 potential. As for the high-lying spectrum, this conclusion may be changed, which needs to be further tested.

It has been shown that the U(4) model can successfully describe nuclear molecules [47, 48]. A nuclear vibron model for nuclear molecules consisting of two clusters holds generally a dynamical symmetry $G_{C1}\otimes G_{C2}\otimes U_R(4)$, where the internal structure of the ith cluster is described by G_{Ci} which may be, for example, the U(6) IBM or SU(3) shell model, and the relative motion between the clusters is described by the $U_R(4)$ vibron model. In some cases, only the U(4) vibron model itself is sufficient to describe the rotational and vibrational excitations in nuclear molecules, where the internal structure of each cluster does not play an essential role in the low-lying levels, such as, the narrow resonances in the ^{12}C+^{12}C system [31, 32]. In the early time, the O(4) limit of U(4) vibron model was proposed to describe the resonant energy of ^{12}C+^{12}C system [31], while the analysis in Ref. [32] indicates that the U(3) limit may be more preferred when fitting the energies of the low-lying resonant states, and Ref. [48] shows that the experimental data are in fact between those with U(3) and O(4) symmetry, respectively. Recently some of us proposed that the ^{12}C+^{12}C system

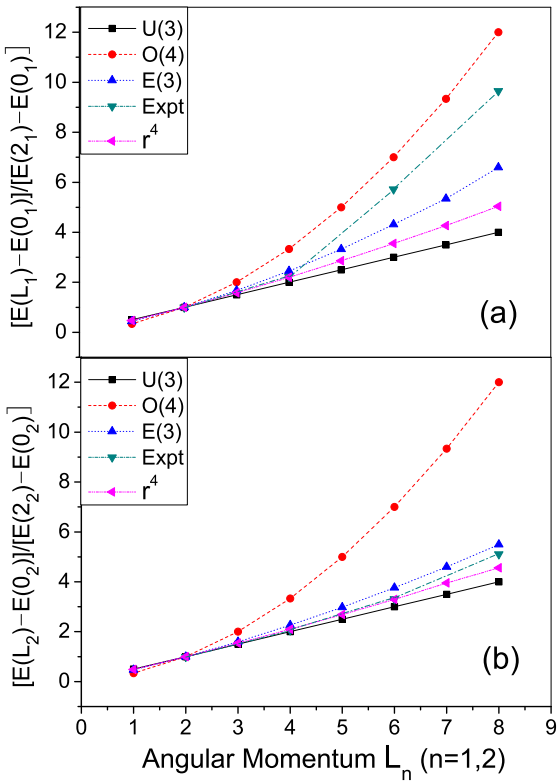

Figure 3: Calculated ratios of the energies of some low-lying states $\frac{E(L_n)-E(0_n)}{E(2_n)-E(0_n)}$ (with $E(0_1) = 0$ and $n = 1$ panel (a), $n = 2$ panel (b)) and some of the experimental data of ^{12}C+^{12}C system.

may be described by the critical symmetry, E(3), in the U(3)-O(4) phase transition [37]. Since the potential at the critical point of the U(3)-O(4) phase transition is in fact in the form of r^4 but not that in the E(3) symmetry, to show the practical possibility of such a system being that at the critical point of the phase transition in the U(4) model, we should re-analyze that more cautiously. Then we display the experimental data of some ratios of the low-lying resonant energies of the ^{12}C+^{12}C system as well as the corresponding results with U(3), O(4), E(3) symmetry and r^4-potential in Table 2 and Fig. 3. One can find from Table 2 and Fig. 3 that the experimental data agree globally better with the results in r^4-potential than with those in the U(3), O(4) and E(3) symmetries. It indicates that, concerning energies of the low-lying resonant states, the ^{12}C+^{12}C system is an empirical evidence of the states at the critical point of the U(3)-O(4) phase transition in the U(4) model, and the E(3) symmetry can describe that approximately.

In summary, we have calculated the energy spectra and the transition rates of not only E1 but also E2 transitions of the low-lying states at the critical point of the phase transition from U(3) to O(4) in the U(4) vibron model in coordinate space with both the E(3) symmetry

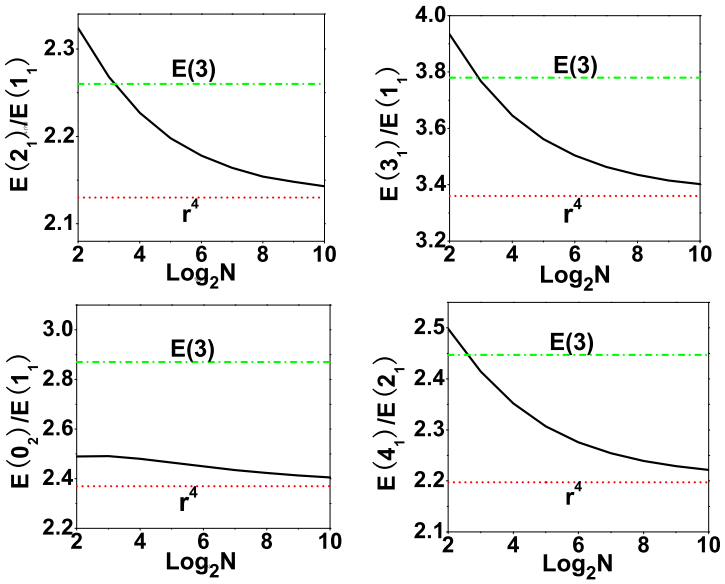

Figure 4: Calculated variation behavior of the energy ratios of some low-lying states with respect to the boson number and those in the E(3) symmetry and in the r^4 potential.

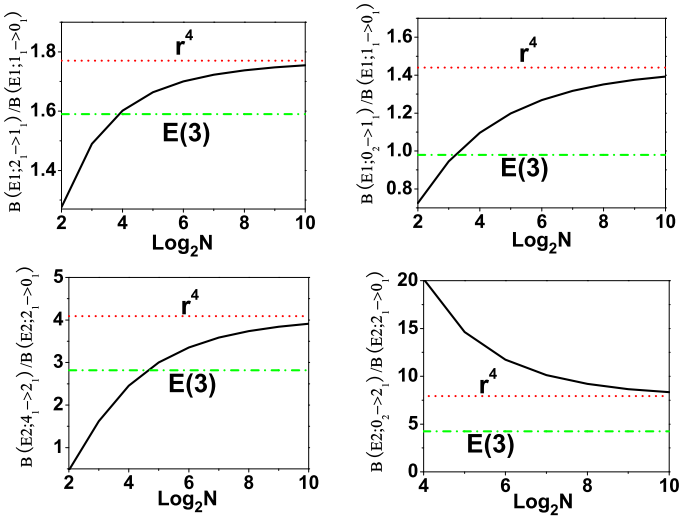

Figure 5: Calculated variation behavior of the ratios of some E1, E2 transition rates between low-lying states with respect to the boson number and those in the E(3) symmetry and in the r^4 potential.

(or with potential in the form of infinite well) and the r^4 potential as well as in boson space with finite N. Our calculation shows that the large-N limit of the U(4) vibron model at the critical point of phase transition can be represented by a r^4 potential model excellently, but

Table 2: Experimental data of the energy ratios of some low-lying resonant states of ^{12}C+^{12}C and the corresponding values in the r^4-potential, E(3) symmetry, U(3) symmetry and O(4) symmetry (the experiment data are taken from Ref. [48]).

	Expt.	E(3)	r^4	U(3)	O(4)
E_{4_1}/E_{2_1}	2.22	2.45	2.20	2	$\frac{10}{3}$
E_{6_1}/E_{2_1}	5.72	4.31	3.55	3	7
E_{0_2}/E_{2_1}	0.31	1.27	1.05	1	$\frac{2N}{3}$
E_{2_2}/E_{2_1}	2.50	3.12	2.39	2	$\frac{2N}{3}+1$
E_{4_2}/E_{2_1}	4.48	5.45	3.79	3	$\frac{2N}{3}+\frac{10}{3}$
E_{6_2}/E_{2_1}	7.55	8.22	5.31	4	$\frac{2N}{3}+7$
E_{0_3}/E_{2_1}	1.86	3.38	2.47	2	$\frac{4N-4}{3}$
E_{4_1}/E_{0_2}	7.16	1.93	1.97	2	$\frac{5}{N}$
E_{6_1}/E_{0_2}	18.45	3.39	3.19	3	$\frac{21}{2N}$
E_{4_1}/E_{0_3}	1.19	0.72	0.89	1	$\frac{5}{2N-2}$
E_{6_1}/E_{0_3}	3.08	1.27	1.43	1.5	$\frac{21}{4N-4}$

not close to those of an infinite well in the $E(3)$ symmetry. Generally, The E(3) symmetry over-predicts the energy levels and under-predicts the E1 and E2 transition rates of the states at the critical point of the phase transition in the U(4) model in large-N limit, but predicts a critical-point spectrum that is qualitatively similar to the U(4) vibron model for small values of N in the most cases. The calculated results also express that the ratios of quantities at the critical point quickly approach to a constant with the increasing of boson number N when $N > 30$, which further confirms the N scaling behavior of quantities at the critical point shown in Ref. [37], where the results indicate that the ratios of both energies and E1 transition at the critical point are approximately invariant as N varies. Comparing the theoretical results with the experimental data of some systems, we find that the ^{12}C+^{12}C system is an empirical evidence of the state at the critical point of the U(3)-O(4) phase transition in the U(4) model and the E(3) symmetry can describe that roughly when concerning the energies of the low-lying resonant states. To conform it much more solidly, one needs the data of electromagnetic transition rates.

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